Problem Tutorial: "Flip"

Consider strings of length $2n$ with n letters `A` and n letters `B`, corresponding to team assignments. What is the probability that a string s corresponds to the final team assignment? Let's define l_A be the position of the last occurrence of `A`, and l_B similarly. Then the probability $p(s) = 2^{-min(l_A, l_B)}$.

We need to find the total probability of strings such that $s_{a_1} = s_{a_2} = \ldots = s_{a_k} = \texttt{A}$.

Let's classify strings on the value of $m = min(l_A, l_B)$ (all such strings have the same probability).

If $m = a_k$, then $s_m = \texttt{A}$ and the number of such strings is $\binom{m-k}{n-k}$.

If $a_i < m < a_{i+1}$ or $m < a_1$ (then let $i = 0$) or $m > a_k$, then $s_m = \texttt{B}$ (in the $m > a_k$ case, this is not the only option) and the number of such strings is $\binom{m-i}{n-1}$. If we find prefix sums of values $\binom{j}{n-1} \cdot 2^{-j}$, we can answer such queries in $O(1)$.

If $m > a_k$, then $s_m = \texttt{A}$ is also possible, and the number of such strings is $\binom{m-k-1}{n-k-1}$. If we find prefix sums of values $\binom{j}{n-k-1} \cdot 2^{-j}$ for each k appearing in the input, we can answer such queries in $O(1)$. There are only $O(\sqrt{n})$ different values of k.

Overall time complexity is $O(n\sqrt{n})$.